

A note on the alternating sums of powers of consecutive q -integers

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Abstract In this paper we construct a new q -Euler numbers and polynomials. By using these numbers and polynomials, we give the interesting formulae related to alternating sums of powers of consecutive q -integers following an idea due to Euler.

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1. Introduction

The Bernoulli numbers among the most interesting and important number sequence in mathematics. They appeared in the posthumous work “ARS Conjectandi (1713)” by Jacob Bernoulli(1654-1705) in connection with sums of powers of consecutive integers(1713). Let n, k be positive integers, and let $S_{n,q}(k)$ be the sums of the n th powers of positive integers up to $k-1$: $S_n(k) = \sum_{l=0}^{k-1} l^n$. Then it was known that the sums of powers of consecutive integers due to J. Bernoulli as follows:

$$S_n(k) = \frac{1}{n+1} \sum_{i=0}^n \binom{n+1}{i} B_i k^{n+1-i}, \text{ where } B_n \text{ are the } n\text{th Bernoulli numbers, cf. [7, 10, 15].}$$

In [1], Carlitz has introduced an interesting q -analogue of Bernoulli numbers, $\beta_{k,q}$. He has indicated a corresponding Stadudt-Clausen theorem and also some interesting congruence properties of the q -Bernoulli numbers. Let q be an indeterminate which can be considered in the complex number field, and for any integer k define the q -integer as $[k]_q = \frac{1-q^k}{1-q}$, cf. [2, 5, 11]. Note that $\lim_{q \rightarrow 1} [k]_q = k$. For any positive integers n, k , let $S_{n,q}(k) = \sum_{l=0}^{k-1} q^l [l]_q^n$, sf. [7, 10]. Then we evaluated sums of powers of consecutive q -integers as follows:

$$S_{n,q}(k) = \frac{1}{n+1} \sum_{i=0}^n \binom{n+1}{i} \beta_{i,q} q^{ki} [k]_q^{n+1-i} - \frac{(1-q^{(n+1)k}) \beta_{n+1,q}}{n+1}, \text{ cf. [7, 10].}$$

The ordinary Euler numbers are defined by

$$\frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{cf. [2, 3, 5, 6, 7, 8],}$$

where we use the usual convention about replacing E^n by $E_n (n > 0)$ symbolically. Let n, k be positive integers, and let $T_{n,q}(k)$ be the alternating sums of the n th powers of positive integers up to $k-1$: $T_n(k) = \sum_{l=0}^{k-1} (-1)^l l^n$. Then Euler investigated the below formulas:

$$T_n(k) = \frac{(-1)^{n+1}}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l k^{n-1} + \frac{E_n}{2} (1 + (-1)^{k+1}).$$

Let u be algebraic in complex number field. Then Frobenius-Euler numbers are defined by

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \quad \text{cf. [3, 5],}$$

note that $H_n(-1) = E_n$, cf. [2, 3, 5, 6, 7]. Carlitz has also introduced an interesting q -analogue of Frobenius-Euler numbers in [1]. A recent author's study of more general q -Euler numbers are found in previous publication [14]. In [4] we gave the new construction of q -Euler numbers, $E_{n,q}^*$, which are different than Carlitz's q -extension and author's q -extension in previous publication (see [11]). Let \mathbb{Z}_p be the ring of p -adic integers, and let p be a fixed odd prime number. Then the p -adic q -integral was defined by author as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q,$$

where $f \in UD(\mathbb{Z}_p)$, cf. [5, 8, 9, 10, 11]. The above q -extension of Euler numbers, $E_{n,q}^*$, were written by

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q}^* \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t}, \quad (\text{see [4, 6]}). \end{aligned} \quad (1)$$

By (1), we easily see that

$$E_{n,q}^* = [2]_q \left(\frac{1}{1-q} \right)^n \sum_{l=0}^n \binom{n}{l} (-1)^l \frac{1}{1+q^{l+1}}, \quad (\text{see [4, 6]}).$$

For $n, m \in \mathbb{N}$, we gave the below interesting formula:

$$\sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m = \frac{1}{[2]_q} ((-1)^{n+1} q^n E_{m,q}^*(n) + E_{m,q}^*), \quad \text{see [4].}$$

In this paper we consider a new approach to q -Euler numbers and polynomials and give some identities and properties between q -Euler numbers and polynomials. Finally we will evaluate the value of $\sum_{l=0}^{n-1} (-1)^l [l]_q^m$ by using our new q -Euler numbers and polynomials. This formula seems to be nice.

2. A note on q -Euler numbers and polynomials

For $q \in \mathbb{C}$ with $|q| < 1$, we consider a modified q -extension of Euler numbers as follows:

$$2 \sum_{l=0}^{\infty} (-1)^l e^{[l]_q t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}. \quad (2)$$

From (2), we can derive the below formula ;

$$\begin{aligned}
2 \sum_{l=0}^{\infty} (-1)^l e^{[l]_q t} &= 2 e^{\frac{t}{1-q}} \sum_{l=0}^{\infty} (-1)^l \sum_{j=0}^{\infty} \left(\frac{1}{1-q}\right)^j q^{lj} \frac{t^j}{j!} \\
&= 2 e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left(\frac{1}{1-q}\right)^j (-1)^j \frac{1}{1+q^j} \frac{t^j}{j!} \\
&= 2 \sum_{i=0}^{\infty} \left(\frac{1}{1-q}\right)^i \frac{t^i}{i!} \sum_{j=0}^{\infty} \left(\frac{1}{1-q}\right)^j (-1)^j \frac{1}{1+q^j} \frac{t^j}{j!} \\
&= 2 \sum_{\substack{n=0 \\ n=i+j}}^{\infty} \left(\left(\frac{1}{1-q}\right)^n \sum_{j=0}^n (-1)^j \left(\frac{1}{1+q^j}\right) \frac{n!}{j!(n-j)!} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(2 \left(\frac{1}{1-q}\right)^n \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{1}{1+q^j} \right) \frac{t^n}{n!}. \tag{3}
\end{aligned}$$

Thus we have the following :

Theorem 1. For $n \geq 0$, we have

$$E_{n,q} = 2 \left(\frac{1}{1-q}\right)^n \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{1}{1+q^j}.$$

Note that $\lim_{q \rightarrow 1} E_{n,q} = E_n$.

By simple calculation, it is easy to check that

$$\begin{aligned}
2 e^{t[x]_q} \sum_{l=0}^{\infty} (-1)^l e^{[l]_q q^x t} &= 2 \sum_{l=0}^{\infty} (-1)^l e^{([x]_q + [l]_q q^x) t} \\
&= 2 \sum_{l=0}^{\infty} (-1)^l e^{[x+l]_q t}. \tag{4}
\end{aligned}$$

From (4), we can define the below q -Euler polynomials :

$$2 \sum_{l=0}^{\infty} (-1)^l e^{[x+l]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{5}$$

By (4) and (5), we easily see that

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} &= 2 \sum_{l=0}^{\infty} (-1)^l \sum_{j=0}^{\infty} [x+l]_q^j \frac{t^j}{j!} \\
&= 2 \sum_{l=0}^{\infty} (-1)^l \sum_{n=0}^{\infty} \binom{n}{j} (-1)^j q^{xj} q^{lj} \frac{t^n}{n!} \\
&= 2 \sum_{n=0}^{\infty} \left(\frac{1}{1-q}\right)^n \sum_{j=0}^n \binom{n}{j} (-1)^j q^{xj} \frac{1}{1+q^j} \frac{t^n}{n!}. \tag{6}
\end{aligned}$$

From (4), (5) and (6), we can obtain the following :

Theorem 2. For $n \geq 0$, we have

$$\begin{aligned}
E_{n,q}(x) &= 2 \left(\frac{1}{1-q}\right)^n \sum_{j=0}^n \binom{n}{j} (-1)^j q^{xj} \frac{1}{1+q^j} \\
&= \sum_{k=0}^{\infty} \binom{n}{k} q^{kx} E_{k,q} [x]_q^{n-k}.
\end{aligned}$$

By (2), (4) and (5), we easily see that

$$\begin{aligned}
& \sum_{m=0}^{\infty} ((-1)^{m+1} E_{m,q}(n) + E_{m,q}) \frac{t^m}{m!} \\
&= -2 \sum_{l=0}^{\infty} (-1)^{l+n} e^{[l+n]_q t} + 2 \sum_{l=0}^{\infty} (-1)^l e^{[l]_q t} \\
&= 2 \sum_{l=0}^{n-1} (-1)^l e^{[l]_q t}.
\end{aligned}$$

Therefore we obtain the following theorem:

Theorem 3. *Let m be the positive integers bigger than 1. Then we have*

$$\frac{((-1)^{1+m} E_{m,q}(n) + E_{m,q})}{2} = \sum_{l=0}^{n-1} (-1)^l [l]_q^m. \quad (6.1)$$

Remark. In [4], it was known that

$$\frac{1}{[2]_q} ((-1)^{1+m} E_{m,q}^*(n) + E_{m,q}^*) = \sum_{l=0}^{n-1} (-1)^l q^l [l]_q^m. \quad (6.2)$$

Comparing (6.1) and (6.2), we see that one new formula (6.1) also seems worthwhile and valuable as same as (6.2)

Remark. In [5], we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} &= \frac{2}{e^t + 1} e^{xt} \\
&= \lim_{q \rightarrow 1} \left(2 \sum_{l=0}^{\infty} (-1)^l e^{[x+l]_q t} \right) \\
&= \sum_{n=0}^{\infty} \left(\lim_{q \rightarrow 1} E_{n,q}(x) \right) \frac{t^n}{n!}.
\end{aligned}$$

Hence, we have $\lim_{q \rightarrow 1} E_{n,q}(x) = E_n(x)$, where $E_n(x)$ are called ordinary Euler polynomials.

From (5), we can also derive ($f \in \mathbb{N}$, odd)

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!} &= 2 \sum_{n=0}^{\infty} (-1)^n e^{[x+n]_q t} \\
&= 2 \sum_{n=0}^{\infty} \sum_{a=0}^{f-1} (-1)^{a+nf} e^{[x+a+nf]_q t} \\
&= \sum_{a=0}^{f-1} (-1)^a \left(2 \sum_{n=0}^{\infty} (-1)^n e^{[f]_q [\frac{x+a}{f} + n]_q t} \right) \\
&= \sum_{m=0}^{\infty} \left(\sum_{a=0}^{f-1} (-1)^a E_{m,q^f} \left(\frac{x+a}{f} \right) \right) [f]_q^m \frac{t^m}{m!},
\end{aligned}$$

where f is odd positive integer.

By comparing the coefficients on both sides, we obtain the following theorem.

Theorem 4. *Let f be a positive odd integer. Then we have*

$$[f]_q^n \sum_{a=0}^{f-1} (-1)^a E_{m,q^f} \left(\frac{x+a}{f} \right) = E_{m,q}(x).$$

For $s \in \mathbb{C}$, let us consider the following complex integration :

$$\frac{1}{\Gamma(s)} \int_0^\infty \sum_{l=0}^\infty (-1)^l e^{-[x+l]_q t} dt = \sum_{l=0}^\infty \frac{(-1)^l}{[x+l]_q^s}. \quad (7)$$

Thus, we can define the Euler q -zeta function as follows :

$$\zeta_{E,q}(s, x) = \sum_{n=0}^\infty \frac{(-1)^n}{[n+x]_q^s}, \quad s \in \mathbb{C}. \quad (8)$$

By (5), (7) and (8), we easily see that $\zeta_{E,q}(-n, x) = \frac{1}{2} E_{n,q}(x)$, $n \in \mathbb{N}$.

Remark. Let $E_n(x)$ be the ordinary Euler polynomials . Then we know that

$$\frac{((-1)^{m+1} E_m(n) + E_m)}{2} = \sum_{l=0}^{n-1} (-1)^l l^m, \quad \text{see [12]}. \quad (9)$$

Theorem 3 is the new q -extension of Eq.(9). Let

$$H_q(s, a; F) = \sum_{m \equiv a(F), m > 0} \frac{(-1)^m}{[m]_q^s} = \sum_{n=0}^\infty \frac{(-1)^{a+nF}}{[a+nF]_q^s} = [F]_q^{-s} (-1)^a \zeta_{E,q^F}(s, \frac{a}{F}),$$

where a and F (=odd) are positive integers with $0 < a < F$. Then we have

$$H_q(-n, a; F) = \frac{(-1)^a [F]_q^n E_{n,q^F}(\frac{a}{F})}{2}, \quad n \geq 1.$$

Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$ (=odd). Then we define the generalized q -Euler numbers attached to χ as follows:

$$F_{\chi,q}(t) = 2 \sum_{n=0}^\infty e^{[n]_q t} \chi(n) (-1)^n = \sum_{n=0}^\infty E_{n,\chi,q} \frac{t^n}{n!}.$$

Note that

$$E_{n,\chi,q} = [d]_q^n \sum_{a=0}^{d-1} \chi(a) (-1)^a E_{n,q^d}(\frac{a}{d}).$$

For $s \in \mathbb{C}$, let us define the $q-l$ -function as follows:

$$l_{E,q}(s, \chi) = \sum_{n=1}^\infty \frac{(-1)^n \chi(n)}{[n]_q^s} = \frac{1}{2} \frac{1}{\Gamma(s)} \int_0^\infty F_{\chi,q}(-t) t^{s-1} dt.$$

Then we easily see that $l_{E,q}(-n, \chi) = \frac{1}{2} E_{n,\chi,q}$, ($n \in \mathbb{N}$). For $s \in \mathbb{C}$, it is easy to see that

$$l_{E,q}(s, \chi) = \sum_{a=1}^F \chi(a) H_q(s, a; F).$$

The function $H_q(s, a; F)$ will be called the partial Euler q -zeta function. Finally we suggest the below problem.

Problem. Find the Witt's formula for the q -Euler numbers (E_n, q) , which was defined in this paper. In [5], the Witt's type formula for $E_{n,q}^*$ was given by

$$\begin{aligned} \sum_{m=0}^\infty E_{n,q}^* \frac{t^n}{n!} &= \sum_{n=0}^\infty \int_{\mathbb{Z}_p} [x]_q^n d\mu_{-q}(x) \frac{t^n}{n!} \\ &= [2]_q \sum_{n=0}^\infty (-1)^n q^n e^{[n]_q t}. \end{aligned}$$

By the same method, it seems to be possible that we give the Witt's formula of q -Euler numbers which can be represented by p -adic q -integrals as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n d\mu_{\square}(x) \\ &= 2 \sum_{l=0}^{\infty} (-1)^l e^{[l]_q t}, \quad (\text{cf. [4, 6, 14]}).\end{aligned}$$

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